PDE based Bounds on Optimal Exponents in Active Simple Hypothesis Testing

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***Abstract*—We present general upper and lower bounds on the optimal error exponent for the Active Simple Hypothesis Testing (ASHT) problem. Although ASHT has been extensively studied across diverse fields—including Information Theory, Bandits, and Operations Research—a precise characterization of this optimal exponent has remained elusive. This work combines a recent Partial Differential Equation (PDE) characterization of the optimal exponent by [Vij25] with the comparison principle of viscosity solutions of Hamilton-Jacobi PDEs. This leads to bounds that are parametrized by convex functions, sets and lower Hamiltonian relaxations, a flexibility that generalizes many known results in the literature.**

***Index Terms*—Active Hypothesis Testing, Best Arm Identification, Fixed Budget, Partial Differential Equations.**

I. Introduction

In Active Simple Hypothesis Testing (ASHT), a controller chooses an action $a \in [K]$, and receives a sample observation conditioned on a from the underlying unknown hypothesis H_i. We assume that H_i is from a known family of channels $\mathcal{H}$. The controller can adaptively choose its future actions based on the prior history of observations and actions. After the budget of T samples has been exhausted, it is required to output a hypothesis $\hat{i}_T \in \mathcal{H}$. The controller aims to minimize probability of error P_e it incurs when choosing $\hat{i}_T$. For a fixed budget T, an optimal solution may be obtained through dynamic programming, but it is quite inefficient computationally and uninsightful. Further, it is well-known from prior works such as [AB10], [KTH22] that the error probability decays exponentially in the sample budget T. Hence, a natural goal is to study the optimal error exponent in the rate of decay of P_e as $T \to \infty$. This work, therefore, studies the ASHT problem in a large deviations setting [Dem09].

This problem is motivated by a wide range of applications, including sensor management [Cas97], [AVF11], channel identification [Hay09], and sequential planning in clinical trials [Arm60]. Owing to its fundamental nature, ASHT has been independently studied across multiple disciplines under various names: as Channel Discrimination [Hay09] and Controlled Sensing [NAV13] in Information Theory; Ranking and Selection [GJ04] in Operations Research; and Fixed-Budget Best Arm Identification [AB10], [KTH22] in the bandit literature.

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In this work, we develop tighter bounds on the optimal error exponent achievable by any feasible control strategy compared to prior works like [GJ04], [Hay09], [NAV13]. We achieve this by combining a characterization [Vij25] of the optimal exponent in terms of a Partial Differential Equation (PDE) and stability bounds [Abg94] for such Hamilton-Jacobi-Isaac (HJI) PDEs.

II. Problem Formulation

We study a hypothesis testing problem with m hypotheses, where the set of hypotheses is denoted by $\mathcal{H} = \{H_1, \cdots, H_m\}$. Each hypothesis H_i is assumed to be a single channel, that is: $H_i := \{P_a^i \mid a \in \mathcal{A}\}$. Here $\mathcal{A}$ denotes the common action set of all the hypotheses. At each time step, the agent chooses an action $a \in \mathcal{A}$ (with $|\mathcal{A}| = K$), and "nature" returns samples from the conditional distribution of action a according to the true unknown underlying channel. We assume these conditional distributions have full support over the common finite set $\mathcal{Y}$.

The controller has a sample budget T, and the controller's policy π_T consists of T many *sampling rules*, denoted as π_T^t, for each $t \in [T]$ and a final *decision rule* $\hat{i}_T$. The sampling rule is *causal* and hence, each π_T^t is adapted to the sample filtration $\mathcal{F}_t$. The decision rule $\hat{i}_T \in [m]$ is adapted to the final sample filtration $\mathcal{F}_T$ and is interpreted as the final hypothesis choice of the controller. We denote the set of all such admissible causal controller policies for a horizon T as $\mathbf{\Pi}_T$. $\mathbf{\Pi}$ is the infinite sequence $(\pi_1, \dots, \pi_T, \dots)$ of such policies such that $\pi_T \in \mathbf{\Pi}_T, \forall T \in \mathbb{N}$.

Defintion 1. *The **error probability** of a policy π_T on the channel hypothesis $H_i \in \mathcal{H}$ to be:*

$$P_e(i, \pi_T) := P_{H_i, \pi_T}(\hat{i}_T \neq i).$$

*The **error exponent** of an admissible sequence of controller policies $\Pi \in \mathbf{\Pi}$ for H_i, is defined as*

$$e(H_i, \Pi) = \liminf_{T \to \infty} -\frac{\log(P_e(i, \pi_T))}{T}$$

*and the **minimal** exponent over the hypotheses $\mathcal{H}$ as*

$$e_m(\mathcal{H}, \Pi) = \min_{H_i \in \mathcal{H}} e(H_i, \Pi).$$

*The **minimax-optimal** exponent is then defined as:*

$$e^*(\mathcal{H}) = \max_{\Pi \in \mathbf{\Pi}} \; e_m(\mathcal{H}, \Pi),$$

and a policy sequence Π^* that attains this maxima[1] *is called the **minimax-optimal policy**.*

Our goal is to find tighter upper and lower bounds for $e^*(\mathcal{H})$.

III. Related Work

[Hay09] showed that for the binary testing ($m = 2$) case, the non-adaptive strategy of playing the action that maximally discriminates the hypotheses in Chernoff Information is optimal. [NAV13] obtained, for $m > 2$, the upper bound $e^*(\mathcal{H}) \le \min_{i \neq j} \max_a C(P_a^i, P_a^j)$, where $C(P, Q)$ denotes the Chernoff Information between P and Q. They present an example where adaptive strategies outperform non-adaptive ones when $m = 3$. [KTH22] used an adaptive sequence of data processing inequalities to obtain a family of upper bounds, which they denoted as R_B^{go}, $B \in \mathbb{N}$. Further, they showed that $\lim_{B\to\infty} R_B^{go} = e^*(\mathcal{H})$. They also proposed an optimal policy that was intractable computationally. To side-step this difficulty, they construct a neural network-based algorithm that has good empirical performance but comes with no optimality guarantees.

[Vij25] showed that $e^*(\mathcal{H}) = V(0, 1)$, where the function V is the *viscosity solution* of the Hamilton-Jacobi-Isaacs (HJI) PDE:

$$\partial_t V + H(\nabla V) = 0 \tag{1}$$

with the initial condition $V(x, 0) = x_{(2)}$ [2]. The Hamiltonian H has the form $H = \min_a H_a$. Here H_a is defined as:

$$H_a(p) = \begin{cases} P \log\left(\sum_{y\in\mathcal{Y}} \prod_{i=1}^m (P_a^i(y))^{\frac{p_i}{P}}\right), & \text{if } P > 0 \\ \max_{y\in\mathcal{Y}} \sum_i p_i \log(P_a^i(y)), & \text{if } P \le 0 \end{cases}$$

with $P = \sum_i p_i$. This HJI-PDE is a consequence of interpreting $e^*(\mathcal{H})$ as the value of a certain differential game between the controller and nature, wherein each player tries to optimize the likelihood of error over all sample paths. Further, they obtained error exponent upper bounds by combining the Comparison principle and Hopf formulae for Hamilton-Jacobi PDEs.

In this paper, we build on [Vij25]'s formulation and use techniques from viscosity solution theory of PDEs to obtain various bounds. Our bounds are achieved by either perturbing the initial condition g or the Hamiltonian H by convex and concave approximations. We obtain three families of bounds, one parametrized by any proper convex functions, the second by proper convex sets and the third by lower relaxations of the Hamiltonian. This improved flexibility generalizes prior bounds obtained in the literature (cf. [Hay09], [NAV13], [KTH22], [Vij25]).

[1] A dynamic programming argument shows the existence of Π^*.

[2] $x_{(i)}$ is the i^{th} order statistic of the vector x.

IV. Preliminaries

Consider a generic Cauchy Problem in $\mathbb{R}^m \times (0, \infty)$ of the form

$$\phi_t + H(\nabla\phi) = 0 \tag{2}$$

with an initial condition $\phi(x, 0) = \phi^0(x)$ where H is a Lipschitz continuous function in $\mathbb{R}^m$ and ϕ^0 is a uniformly Lipschitz function. It is known that under these regularity conditions, the above Cauchy problem admits a unique Lipschitz continuous *viscosity solution* [CL83].

The *Comparison Principle* [CL83] states that for any two Cauchy problems for HJI PDEs such that $H_a \ge H_b$ and $\phi_a^0 \le \phi_b^0$, we must have that $\phi_a(x, t) \le \phi_b(x, t), \forall (x, t) \in \mathbb{R}^m \times (0, \infty)$.

The *Hopf and Hopf–Lax* formulae [Eva22] provide closed-form solutions to the HJI PDEs whenever either the initial datum ϕ^0 or the Hamiltonian H is convex or concave. For a **convex** initial function ϕ^0, the solution is given by the Hopf formula:

$$\phi(x, t) = \sup_{v\in\mathbb{R}^m} \Big[x \cdot v - \phi^{0*}(v) - t\, H(v)\Big].$$

Similarly, when ϕ^0 is **concave**, the solution is:

$$\phi(x, t) = \inf_{v\in\mathbb{R}^m} \Big[x \cdot v - \phi^{0*}(-v) - t\, H(v)\Big].$$

Here f^* denotes the appropriate Legendre–Fenchel conjugate [Roc97] of the function f. In the case when Hamiltonian H is **convex**, the solution is given by the Hopf–Lax formula:

$$\phi(x, t) = \inf_{y\in\mathbb{R}^m} \Big[\phi^0(y) + t\, H^*\big(\tfrac{x-y}{t}\big)\Big],$$

and for a **concave** Hamiltonian by:

$$\phi(x, t) = \sup_{y\in\mathbb{R}^m} \Big[\phi^0(y) - t\,(-H)^*\big(\tfrac{x-y}{t}\big)\Big].$$

We aim to obtain upper and lower bounds on the viscosity solution of (1) by approximating $\phi^{(0)}$ or H by concave or convex bounds and applying the comparison principle. The results of [Vij25] then translate these bounds to $e^*(\mathcal{H})$.

The results of [Abg94] are very useful in this endeavour. However, the precise result [3] stated there is incorrect. We give a corrected version of this result below and provide a proof.

Proposition 1 (Corrected result of [Abg94]). *For any convex function f, let $D_f = \{x \in \mathbb{R}^m, f^*(x) < +\infty\}$ and similarly for a concave function let it denote $D_f = \{x \in \mathbb{R}^m, f^*(x) > -\infty\}$. For any initial datum $\phi^0 = \phi_1^0 + \phi_2^0$, where ϕ_1^0 is convex and ϕ_2^0 is concave, we have that*

$$\inf_{v\in D_{\phi_2^0}} \sup_{w\in\mathbb{R}^m} \Big\{x \cdot w - \phi_2^{0*}(v) - \phi_1^{0*}(v + w) - tH(w)\Big\}$$
$$\ge \phi(x, t) \ge$$
$$\sup_{v\in D_{\phi_1^0}} \inf_{w\in\mathbb{R}^m} \Big\{x \cdot w - \phi_1^{0*}(v) - \phi_2^{0*}(v - w) - tH(w)\Big\}$$

[3] Proposition 1.1 in [Abg94].

Proof. Since $\phi_1^{0*}(v) = \sup_{x\in\mathbb{R}^m}[x\cdot v - \phi_1^0(x)]$, we have that

$$\phi_1^0(x) \geq x\cdot v - \phi_1^{0*}(v)$$
$$\phi^0(x) \geq x\cdot v + \phi_2^0(x) - \phi_1^{0*}(v) := \xi_v(x)$$

Clearly, $\xi_v(x)$ is a concave function of x. The Hopf formula for a Cauchy Problem with initial datum as ξ_v gives its corresponding viscosity solution $\xi_v(x,t)$ as:

$$\xi_v(x,t) = \inf_{w\in\mathbb{R}^m}\Bigg\{x\cdot w - \xi_v^*(-w) - tH(w)\Bigg\}.$$

We observe that:

$$\begin{aligned}\xi_v^*(-w) &= \inf_{x\in\mathbb{R}^m}\Bigg\{+x\cdot w - \xi_v(x)\Bigg\}\\ &= \inf_{x\in\mathbb{R}^m}\Bigg\{x\cdot(w-v) + \phi_1^{0*}(v) - \phi_2^0(x)\Bigg\}\\ &= \inf_{x\in\mathbb{R}^m}\Bigg\{-x\cdot(v-w) + \phi_1^{0*}(v) - \phi_2^0(x)\Bigg\}.\end{aligned}$$

Thus, we have:

$$\xi_v^*(-w) = \begin{cases}+\infty, & v\notin D_{\phi_1^0}\\ \phi_2^{0*}(v-w) + \phi_1^{0*}(v) & v\in D_{\phi_1^0}.\end{cases}$$

Plugging this back into the Hopf representation, we obtain, $\forall v\in D_{\phi_1^0}$ that:

$$\xi_v(x,t) = \inf_{w\in\mathbb{R}^m}\Bigg\{x\cdot w - \phi_2^{0*}(v-w) - \phi_1^{0*}(v) - tH(w)\Bigg\}.$$

Using the comparison principle we get, for any $v\in D_{\phi_1^0}$

$$\phi(x,t) \geq \xi_v(x,t) = \inf_{w\in\mathbb{R}^m}\Bigg\{x\cdot w - \phi_2^{0*}(v-w) - \phi_1^{0*}(v) - tH(w)\Bigg\},$$

and by supremizing over all feasible v we get the lower bound:

$$\phi(x,t) \geq \sup_{v\in D_{\phi_1^0}}\inf_{w\in\mathbb{R}^m}\Bigg\{x\cdot w - \phi_1^{0*}(v) - \phi_2^{0*}(v-w) - tH(w)\Bigg\}.$$

We repeat the arguments for ϕ_2^0 to obtain an upper bound. Since $\phi_2^{0*}(w) = \inf_{x\in\mathbb{R}^m}[x\cdot w - \phi_2^0(x)]$, we get that:

$$\phi^0(x) \leq x\cdot w + \phi_1^0(x) - \phi_2^{0*}(w) := \eta_w(x).$$

Again applying Hopf formula since η_w is convex function gives us a representation for the viscosity solution $\eta_w(x,t)$:

$$\eta_w(x,t) = \sup_{v\in\mathbb{R}^m}\Bigg\{x\cdot v - \eta_w^*(v) - tH(v)\Bigg\}.$$

Again, as above, we have that:

$$\eta_w^*(v) = \begin{cases}-\infty, & w\notin D_{\phi_2^0}\\ \phi_2^{0*}(w) + \phi_1^{0*}(w-v) & w\in D_{\phi_2^0}.\end{cases}$$

Using the comparison principle and infimizing over all feasible w gives:

$$\phi(x,t) \leq \inf_{w\in D_{\phi_2^0}}\sup_{v\in\mathbb{R}^m}\Bigg\{x\cdot w - \phi_2^{0*}(w) - \phi_1^{0*}(w-v) - tH(v)\Bigg\}.$$

□

V. Bounds on the solution of (1).

A. Bounds using Abgrall's analysis

To apply Proposition 1 to HJI-PDE (1), we start by decomposing the initial condition $x_{(2)}$ as a sum of convex and concave functions.

Lemma 1.

$$x_{(2)} = \sigma_P(x) - \sigma_Q(x)$$

where σ_P, σ_Q represent the support functions[4] *of the convex bodies*

$$P = \text{convhull}(\{\mathbf{1} - e_i\}_i),\ Q = \text{convhull}(\{\mathbf{1} - e_i - e_j\}_{i,j:i\neq j})$$

respectively. Here $\{e_i\}_{i\in[m]}$ is the standard basis in $\mathbb{R}^m$.

Proof. Since

$$\begin{aligned}x_{(2)} &= \Bigg[\sum_i x_i - \min_i\{x_i\}\Bigg] + \Bigg[-\max_{|S|=n-2}\sum_{i\in S} x_i\Bigg]\\ &= \sup_{\lambda\in P}\langle\lambda, x\rangle - \sup_{\mu\in Q}\langle\mu, x\rangle,\end{aligned}$$

we have that:

$$x_{(2)} = \sigma_P(x) - \sigma_Q(x).$$

□

Fact 1. *For a convex body P, the Legendre-Fenchel conjugate of its support function satisfies*

$$\sigma_P^*(x) = \delta_P(x)$$

where $\delta_P(x) = 0$ for $x\in P$ and $+\infty$ otherwise.

See standard convex analysis texts like [Roc15], [HUL96] for a proof of this. Using this sum of convex and concave functions decomposition, one obtains the following upper and lower bounds:

Theorem 1.

$$\begin{aligned}&\inf_{v\in Q}\sup_{u\in P}\Big\{x\cdot(u-v) - tH(u-v)\Big\}\\ &\geq V(x,t) \geq\\ &\sup_{v\in P}\inf_{u\in Q}\Big\{x\cdot(v-u) - tH(v-u)\Big\}\end{aligned}$$

And, since, $e^*(\mathcal{H}) = V(0,1)$ satisfies

$$\begin{aligned}&\inf_{v\in Q}\sup_{u\in P}\Big\{-H(u-v)\Big\}\\ &\geq e^*(\mathcal{H}) \geq\\ &\sup_{v\in P}\inf_{u\in Q}\Big\{-H(v-u)\Big\}\end{aligned}$$

[4] $\sigma_R(x) := \sup_{y\in R}\langle y, x\rangle$, see for example [Roc97].

Proof. This is a direct application of Proposition 1. Use Fact 1 with the decomposition $x_{(2)} = \sigma_P(x) - \sigma_Q(x)$. $\square$

Remark 1. *In the upper bound in Theorem 1, we can show that for any choice of $v \in Q$, there exists a choice of $u \in P$, so that $u - v \in \Delta_m$. If it was true for any choice of $v \in Q$, the maximum is always attained when $u - v \in \Delta_m$, then Theorem 1 improves on Proposition 4.11 of [Vij25] [5]. We are currently unable to show this and leave it for future work.*

Using Theorem 1, one can get another class of bounds. This is done by noting that the sum of convex and concave function decomposition is not unique. Let $\mathscr{C}(\mathbb{R}^m)$ be the set of *all* convex functions on $\mathbb{R}^m$ and $\mathscr{C}_P(\mathbb{R}^m)(\subset \mathscr{C}(\mathbb{R}^m))$ be the subset of all proper, convex, and lower semi-continuous functions. To begin, let $F \in \mathscr{C}_P(\mathbb{R}^m)$. Consider the initial function decomposition:

$$x_{(2)} = \tilde{\phi}_1^0 + \tilde{\phi}_2^0 = [\sigma_P(x) + F^*(x)] - [\sigma_Q(x) + F^*(x)]$$

.

Fact 2. *Let $f, g \in \mathscr{C}(\mathbb{R}^m)$ be proper, convex and lower semi-continuous. Then*

$$[f+g]^* = (f^* \,\square\, g^*),$$

where $\square$ denotes the infimal convolution:

$$f \square g(x) := \inf\{f(r) + g(s) | r + s = x\},$$

between two convex functions f and g.

Again, the reader should look at standard convex analysis books like [Roc15] for a proof of the above fact.

Theorem 2. *Let F be a proper, convex and lower semi-continuous function. Then,*

$$\inf_{v \in D_{\tilde{\phi}_2^0}} \sup_{z \in \mathbb{R}^m} \Big\{ (\delta_Q \,\square\, F)(v) - (\delta_P \,\square\, F)(z) - H(z - v) \Big\}$$
$$\geq e^*(\mathcal{H}) \geq$$
$$\sup_{v \in D_{\tilde{\phi}_1^0}} \inf_{z \in \mathbb{R}^m} \Big\{ -(\delta_P \,\square\, F)(v) + (\delta_Q \,\square\, F)(z) - H(z - v) \Big\}.$$

Proof. We have

$$x_{(2)} = [\sigma_P(x) + F^*(x)] - [\sigma_Q(x) + F^*(x)] = \tilde{\phi}_1^0 + \tilde{\phi}_2^0.$$

By Fact 2,

$$\tilde{\phi}_1^{0*}(z) = [\sigma_P + F^*]^*(z) = (\delta_P \,\square\, F)(z)$$

and similarly

$$-\tilde{\phi}_2^{0*}(v) = (-\tilde{\phi}_2^0)^*(v) = [\sigma_Q + F^*]^*(v) = (\delta_Q \,\square\, F)(v).$$

Now, applying Theorem 1 one gets the result. $\square$

We also have the following result by optimizing the above over $F \in \mathscr{C}_P(\mathbb{R}^m)$:

[5] [Vij25] already improves on [Hay09], [NAV13].

Corollary 1.

$$\inf_{\substack{F \in \mathscr{C}_P(\mathbb{R}^m) \\ v \in D_{\tilde{\phi}_2^0}}} \sup_{z \in \mathbb{R}^m} \Big\{ (\delta_Q \square F)(v) - (\delta_P \square F)(z) - H(z - v) \Big\}$$
$$\geq e^*(\mathcal{H}) \geq$$
$$\sup_{\substack{F \in \mathscr{C}_P(\mathbb{R}^m) \\ v \in D_{\tilde{\phi}_1^0}}} \inf_{z \in \mathbb{R}^m} \Big\{ -(\delta_P \square F)(v) + (\delta_Q \square F)(z) - H(z - v) \Big\}$$

The bounds established have a distinct game-theoretic character to them. In the lower bounds, one wishes to make the supremizing player have more choices while lessening the choices of the infimizing player. The opposite considerations apply for the upper bound. In Theorem 1, the choices are hard constrained to the sets P, Q, while in the results Theorem 2 and Corollary 1, the choices are only soft constrained by distances to P, Q as measured by F.

We will now derive the bounds for some specific functions $F \in \mathscr{C}_P(\mathbb{R}^m)$. We study the simple classes of support functions and squared Euclidean distances.

Corollary 2. *For R a convex body take $F = \sigma_R$,*

$$\inf_{v \in D_{\tilde{\phi}_2^0}} \sup_{z \in \mathbb{R}^m} \Big\{ (\delta_Q \,\square\, \sigma_R)(v) - (\delta_P \,\square\, \sigma_R)(z) - H(z - v) \Big\}$$
$$\geq e^*(\mathcal{H}) \geq$$
$$\sup_{v \in D_{\tilde{\phi}_1^0}} \inf_{z \in \mathbb{R}^m} \Big\{ -(\delta_P \,\square\, \sigma_R)(v) + (\delta_Q \,\square\, \sigma_R)(z) - H(z - v) \Big\}$$

This gives a bound based on arbitrary convex sets R.

Corollary 3. *For $F(x) = \|x - c\|_2^2$,*

$$\inf_{v \in D_{\tilde{\phi}_2^0}} \sup_{z \in \mathbb{R}^m} \Big\{ \mathrm{d}^2(v,\, Q + c) - \mathrm{d}^2(z,\; P + c) - H(z - v) \Big\}$$
$$\geq e^*(\mathcal{H}) \geq$$
$$\sup_{v \in D_{\tilde{\phi}_1^0}} \inf_{z \in \mathbb{R}^m} \Big\{ -\mathrm{d}^2(v,\; P + c) + \mathrm{d}^2(z,\; Q + c) - H(z - v) \Big\}$$

where $\mathrm{d}^2(x, C)$ denotes the euclidean distance of x from a set C.

In Corollary 3, choosing the distance function penalizes the players from moving too far away from the sets P and Q, but the penalization is not as strong as $F \equiv 0$. The hope is to find a sweet spot.

One can write $-H(p) = \max_a(-H_a(p))$, switch $\inf, \max$ and obtain the exact infima using the fact that

$$\nabla_v \,\mathrm{d}^2(v, Q + c) = 2(v - \pi_{Q+c}(v))$$

where $\pi_R(v) = \arg\min_{q \in R} \|v - q\|_2^2$.

Another natural choice for F is δ_R. However, one can easily prove, for that choice of F, the bound obtained would be necessarily worse than Theorem 1. We expect the tightest bounds to come from when F is chosen to respect the natural geometry of the ASHT problem. One such choice is $F(.) = \sum_a w_a H_a(.)$, with $w \in \Delta_m$. This leads to considerations involving I-projections [Nie18] on the sets P, Q. We intend to pursue this approach in future work.

B. Bounds using a Hamiltonian relaxation

In this section, we will obtain bounds by approximating the Hamiltonian. To this end, let $w = (w_a)_{a\in\mathcal{A}} \in \Delta_K$ i.e. $\sum_{a\in A} w_a = 1, w_a \geq 0$. Define $\tilde{H}_w(p) = \sum_{a\in A} w_a H_a(p)$. Since $H_a(\cdot)$ is convex, so is $\tilde{H}_w(\cdot)$. We then have that:

$$H^{**} \leq H \leq \tilde{H}_w. \tag{3}$$

We use these inequalities to bound $e^*(\mathcal{H})$ using the Hopf-Lax formula.

Theorem 3 (Hamiltonian Relaxation)**.** *We have that:*

$$\min_{y\in\mathbb{R}^m} \sup_{w\in\Delta_K} \left\{ t\tilde{H}_w^* \left(\frac{x-y}{t}\right) + y_{(2)} \right\} \geq V(x,t) \geq \sup_{w\in\Delta_K} \min_{y\in\mathbb{R}^m} \left\{ t\tilde{H}_w^* \left(\frac{x-y}{t}\right) + y_{(2)} \right\}$$

And, in particular,

$$\min_{y\in\mathbb{R}^m} \sup_{w\in\Delta_K} \left\{ \tilde{H}_w^* (-y) + y_{(2)} \right\} \geq e^*(\mathcal{H}) \geq \sup_{w\in\Delta_K} \min_{y\in\mathbb{R}^m} \left\{ \tilde{H}_w^*(-y) + y_{(2)} \right\}$$

Proof. Consider a new Cauchy problem (2) with the relaxed Hamiltonian $\tilde{H}_w(\cdot)$ and the same initial datum $y_{(2)}$. Denote the viscosity solution of this problem as $\tilde{V}_w$, that is,

$$\partial_t \tilde{V}_w + \tilde{H}_w(\nabla \tilde{V}_w) = 0 \tag{4}$$

with $\tilde{V}_w(x,0) = x_{(2)}$.

Applying the Comparison Principle with the inequality $H \leq \tilde{H}_w$ and then invoking Hopf-Lax, we then get:

$$V(x,t) \geq \tilde{V}_w(x,t) = \min_{y\in\mathbb{R}^m} \left\{ t\tilde{H}_w^* \left(\frac{x-y}{t}\right) + y_{(2)} \right\}$$

Supremizing over the allocation $w = (w_a)_{a\in\mathcal{A}}$ we get

$$V(x,t) \geq \sup_{w\in\Delta_K} \min_{y\in\mathbb{R}^m} \left\{ t\tilde{H}_w^* \left(\frac{x-y}{t}\right) + y_{(2)} \right\}$$

which proves the lower bound. For the upper bound, we use the other inequality $H^{**} \leq H$ in (3) and combine as above with the Hopf-Lax formula. The upper-bound:

$$\min_{y\in\mathbb{R}^m} \left\{ tH^* \left(\frac{x-y}{t}\right) + y_{(2)} \right\} \geq V(x,t),$$

then follows from the generic fact in analysis that for any continuous function f, $f^{***} = f^*$. The bounds on $e^*(\mathcal{H})$ are obtained then from [Vij25]'s result $V(0,1) = e^*(\mathcal{H})$ and observing $\sup_w \tilde{H}_w^* = H^*$. □

Remark 2. *The R_B^{go} upper bounds of [KTH22] was shown by [Vij25] to be just evaluations of the functional $y_{(2)}$ at equilibrium points of a B-stage repeated Stackelberg game. We believe that for any such Stackelberg equilibrium point y_B^*, it is the case that $H^*(-y_B^*) \leq 0$. This would imply that the upper bounds of Theorem 2 improve on the R_B^{go} of [KTH22]. We are unable to show this and leave it for future work.*

VI. Conclusion.

We obtained flexible upper and lower bounds to the minimax-optimal error exponent $e^*(\mathcal{H})$, by utilizing the PDE characterization (1) along with comparison results for viscosity solutions on convex or concave approximations to Hamiltonian and initial conditions of the PDE. The bounds obtained are more general than prior work and are flexibly parametrized by convex functions, sets and lower Hamiltonian relaxations. We conjecture that these bounds are strict improvements over the existing work, like [KTH22], [Vij25].

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